

# PRIME FILTRATIONS OF MONOMIAL IDEALS AND POLARIZATIONS

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**ABSTRACT.** We show that all monomial ideals in the polynomial ring in at most 3 variables are pretty clean and that an arbitrary monomial ideal  $I$  is pretty clean if and only if its polarization  $I^p$  is clean. This yields a new characterization of pretty clean monomial ideals in terms of the arithmetic degree, and it also implies that a multicomplex is shellable if and only if the simplicial complex corresponding to its polarization is (non-pure) shellable. We also discuss Stanley decompositions in relation to prime filtrations.

## INTRODUCTION

Let  $R$  be a Noetherian ring, and  $M$  a finitely generated  $R$ -module. A basic fact in commutative algebra [8, Theorem 6.4] says that there exists a finite filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \dots \subset M_r = M$$

with cyclic quotients  $M_i/M_{i-1} \cong R/P_i$  and  $P_i \in \text{Supp}(M)$ . We call any such filtration of  $M$  a prime filtration. The set of prime ideals  $P_1, \dots, P_r$  which define the cyclic quotients of  $\mathcal{F}$  will be denoted by  $\text{Supp}(\mathcal{F})$ . Another basic fact [8, Theorem 6.5] implies that  $\text{Ass}(M) \subset \text{Supp}(\mathcal{F}) \subset \text{Supp}(M)$ . Let  $\text{Min}(M)$  denote the set of minimal prime ideals in  $\text{Supp}(M)$ . Dress [3] calls a prime filtration  $\mathcal{F}$  of  $M$  *clean* if  $\text{Supp}(\mathcal{F}) = \text{Min}(M)$ . The  $R$ -module  $M$  is called *clean* if it admits a clean filtration. Herzog and Popescu [5] introduced the concept of *pretty clean modules*.

A prime filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \dots \subset M_r = M$$

of  $M$  with  $M_i/M_{i-1} \cong R/P_i$  is called *pretty clean*, if for all  $i < j$  for which  $P_i \subseteq P_j$  it follows that  $P_i = P_j$ .

In other words, a proper inclusion  $P_i \subset P_j$  is only possible if  $i > j$ . The module  $M$  is called *pretty clean*, if it has a pretty clean filtration. We say an ideal  $I \subset R$  is pretty clean if  $R/I$  is pretty clean.

A prime filtration which is pretty clean has the nice property that  $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$ , see [5, Corollary 3.6]. It is still an open problem to characterize the modules which have a prime filtration  $\mathcal{F}$  with  $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$ . In Section 4 we give an example of a module which is not pretty clean but nevertheless has a prime filtration whose support coincides with the set of associated prime ideals of  $M$ .

Dress showed in his paper [3] that a simplicial complex is shellable if and only if its Stanley-Reisner ideal is clean, and Herzog and Popescu generalized this result by showing that the multicomplex associated with a monomial ideal  $I$  is shellable if and only if  $I$  is pretty clean. As a main result of this paper we relate these two results by showing in Theorem 3.10 that a monomial ideal is pretty clean if and only if its polarization  $I^p$  is clean. As a consequence of this result we are able to give the following characterization

(Theorem 4.2) of pretty clean monomial ideals: for any monomial ideal  $I$  the length of any prime filtration is bounded below by the arithmetic degree of  $I$ , and equality holds if and only if  $I$  is pretty clean.

In the first section of this paper we show that all monomial ideals in  $K[x_1, \dots, x_n]$  of height  $\geq n - 1$  are pretty clean and use this fact to show that any monomial ideal in the polynomial ring in three variables is pretty clean; see Proposition 1.7 and Theorem 1.10. However for all  $n \geq 4$  there exists a monomial ideal of height  $n - 2$  which is not pretty clean, see Example 1.11.

In Section 2 we discuss the Stanley conjecture concerning Stanley decompositions. In [5, Theorem 6.5] it was shown that the Stanley conjecture holds for any pretty clean monomial ideal. Therefore using the results of Section 1 we recover the result of Apel [1, Theorem 5.1] that the Stanley conjecture holds for any monomial ideal in the polynomial ring in three variables. Similarly we conclude that the Stanley conjecture holds for any monomial ideal of codimension 1.

We also notice (Proposition 2.2) that for a monomial ideal, instead of requiring that  $I$  is pretty clean, it suffice to require that there exists a prime filtration  $\mathcal{F}$  with  $\text{Ass}(S/I) = \text{Supp}(\mathcal{F})$  in order to conclude that the Stanley conjecture holds for  $S/I$ .

Unfortunately it is not true that each Stanley decomposition corresponds to a prime filtration as shown by an example of MacLagan and Smith [7, Example 3.8]. However we characterize in Proposition 2.7 those Stanley decomposition of  $S/I$  that correspond to prime filtrations. Using this characterization we show in Corollary 2.8 that in the polynomial ring in two variables Stanley decompositions and prime filtrations are in bijective correspondence.

In Section 3 we prove the above mentioned result concerning polarizations. One important step in the proof (see Proposition 3.8) is to show that there is a bijection between the facets of the multicomplex defined by the monomial ideal  $I$  and the facets of the simplicial complex defined by the polarization of this shows that  $I$  and  $I^p$  have the same arithmetic degree.

The final Section is devoted to prove the new characterization of pretty clean monomial ideals in terms of the arithmetic degree.

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## 1. PRETTY CLEAN MONOMIAL IDEALS AND MULTICOMPLEXES

We denote by  $S = K[x_1, \dots, x_n]$  the polynomial ring in  $n$  variables over a field  $K$ . Let  $I \subset S$  be a monomial ideal. In this paper a prime filtration of  $I$  is always assumed to be a monomial prime filtration. This means a prime filtration

$$\mathcal{F} : I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

with  $I_j/I_{j-1} \cong S/P_j$ , for  $j = 1, \dots, r$  such that all  $I_j$  are monomial ideals.

Recall that the prime filtration  $\mathcal{F}$  is called *pretty clean*, if for all  $i < j$  which  $P_i \subseteq P_j$  it follows that  $P_i = P_j$ . The monomial ideal  $I$  is called *pretty clean*, if it has a pretty clean filtration.

In this section we will show that monomial ideals in at most three variables are pretty clean.

Let  $I \subset S$  be a monomial ideal. The saturation  $\tilde{I}$  of  $I$  is defined to be

$$\tilde{I} = I : \mathfrak{m}^\infty = \bigcup_k (I : \mathfrak{m}^k),$$

where  $\mathfrak{m} = (x_1, \dots, x_n)$  is the graded maximal ideal of  $S$ .

We first note the following

**Lemma 1.1.** *Let  $I \subset S$  be a monomial ideal of  $S$ . Then  $I$  is pretty clean if and only if  $\tilde{I}$  is pretty clean.*

*Proof.* The  $K$ -vectorspace  $\tilde{I}/I$  has a finite dimension, and we can choose monomials  $u_1, \dots, u_t \in \tilde{I}$  whose residue classes modulo  $I$  form a  $K$ -basis of  $\tilde{I}/I$ . Moreover the basis can be chosen such that for all  $j = 1, \dots, t$  one has  $I_j/I_{j-1} \cong S/\mathfrak{m}$  where  $I_0 = I$  and  $I_j = (I_{j-1}, u_j)$ , and where  $\mathfrak{m} = (x_1, \dots, x_n)$  is the graded maximal ideal of  $S$ . Indeed, we have  $\tilde{I} = I : \mathfrak{m}^k$  for some  $k$ . For each  $i \in [k]$ , where  $[k] = \{1, \dots, k\}$ , the  $K$ -vectorspace  $(I : \mathfrak{m}^i)/(I : \mathfrak{m}^{i-1})$  has finite dimension. If

$$\dim_K(I : \mathfrak{m}^i / I : \mathfrak{m}^{i-1}) = r_i,$$

then we can choose monomials  $u_{i1}, \dots, u_{ir_i} \in I : \mathfrak{m}^i$  whose residue classes modulo  $I : \mathfrak{m}^{i-1}$  form a basis for this  $K$ -vectorspace. Composing these bases we obtain the required basis for  $\tilde{I}/I$ .

So we have

$$\mathcal{F}_1 : I = I_0 \subset I_1 \subset \dots \subset I_t = \tilde{I}$$

with  $I_i/I_{i-1} \cong S/\mathfrak{m}$ , for all  $i = 1, \dots, t$ . Now if  $\tilde{I}$  is a pretty clean and  $\mathcal{G}$  is pretty the clean filtration of  $\tilde{I}$ , then the prime filtration  $\mathcal{F}$  which is obtained by composing  $\mathcal{F}_1$  and  $\mathcal{G}$  yields a pretty clean filtration of  $I$ .

For the converse, let  $I = I_0 \subset I_1 \subset \dots \subset I_r = S$  be pretty clean filtration of  $I$ . We will show that  $\tilde{I}$  is pretty clean by induction on  $\dim_K \tilde{I}/I = t$ . If  $t = 0$  the assertion is trivially true. Assume now that  $t > 0$ . It is clear that  $I_1$  is also pretty clean and that  $I_1/I \cong S/\mathfrak{m}$ , since  $I \neq \tilde{I}$ . It follows that  $\tilde{I}_1 = \tilde{I}$  and that  $\dim_K \tilde{I}_1/I_1 = t - 1$ . So by the induction hypothesis  $\tilde{I} = \tilde{I}_1$  is pretty clean.  $\square$

**Corollary 1.2.** *Let  $S = K[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables. Then any monomial ideal of height  $n$  is pretty clean.*

Our next goal is to show that even the monomial ideals in  $S = K[x_1, \dots, x_n]$  of height  $\geq n - 1$  are pretty clean. To this end we have to recall the concept of multicomplexes and shellings.

Stanley [11] calls a subset  $\Gamma \subseteq \mathbb{N}^n$  a *multicomplex* if for all  $a \in \Gamma$  and for all  $b \leq a$  one has  $b \in \Gamma$ . Herzog and Popescu [5] give the following modification of Stanley's definition of multicomplex which will be used in this paper. Before we give this definition we introduce some notation. We set  $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ . Let  $\Gamma$  be a subset of  $\mathbb{N}_\infty^n$ . An element  $m \in \Gamma$  is called maximal if there is no  $a \in \Gamma$  with  $a > m$ . We denote by  $M(\Gamma)$  the set of maximal elements of  $\Gamma$ . If  $a \in \Gamma$ , we call

$$\text{infpt}(a) = \{i : a(i) = \infty\},$$

the *infinite part* of  $a$ .

**Definition 1.3.** A subset  $\Gamma \subset \mathbb{N}_\infty^n$  is called a *multicomplex* if

- (i) for all  $a \in \Gamma$  and for all  $b \leq a$  it follows that  $b \in \Gamma$ ,
- (ii) for all  $a \in \Gamma$  there exists an element  $m \in M(\Gamma)$  such that  $a \leq m$ .

The elements of a multicomplex are called *faces*. An element  $a \in \Gamma$  is called a *facet* of  $\Gamma$  if for all  $m \in M(\Gamma)$  with  $a \leq m$  one has  $\text{infpt}(a) = \text{infpt}(m)$ . The set of all facets of  $\Gamma$  will be denoted by  $F(\Gamma)$ . The facets in  $M(\Gamma)$  are called *maximal facets*. It is clear that  $M(\Gamma) \subset F(\Gamma)$ . We recall that for each multicomplex  $\Gamma$  the set of facets of  $\Gamma$  is a finite set, see [5, Lemma 9.6].

Let  $\Gamma$  be a multicomplex, and let  $I(\Gamma)$  be the  $K$ -vectorspace in  $S = K[x_1, \dots, x_n]$  spanned by all monomials  $x^a$  such that  $a \notin \Gamma$ . Note that  $I(\Gamma)$  is a monomial ideal, and called the monomial ideal associated to  $\Gamma$ .

Conversely let  $I \subset S$  be any monomial ideal, then there exists a unique multicomplex  $\Gamma(I)$  with  $I(\Gamma(I)) = I$ . Indeed, let  $A = \{a \in \mathbb{N}^n : x^a \notin I\}$ ; then  $\Gamma(I) = \Gamma(A)$  is called the multicomplex associated to  $I$ , where  $\Gamma(A) = \{b \in \mathbb{N}_\infty^n : b \leq a \text{ for some } a \in A\}$ .

A subset  $S \subset \mathbb{N}_\infty^n$  is called a *Stanley set* if there exists  $a \in \mathbb{N}^n$  and  $m \in \mathbb{N}_\infty^n$  with  $m(i) \in \{0, \infty\}$  such that  $S = a + S^*$ , where  $S^* = \Gamma(m)$ .

In [5] the concept of *shelling* of multicomplexes was introduced as in the following by Herzog and Popescu.

**Definition 1.4.** A multicomplex  $\Gamma$  is *shellable* if the facets of  $\Gamma$  can be ordered  $a_1, \dots, a_r$  such that

- (i)  $S_i = \Gamma(a_i) \setminus \Gamma(a_1, \dots, a_{i-1})$  is a Stanley set for all  $i = 1, \dots, r$ , and
- (ii) whenever  $S_i^* \subseteq S_j^*$ , then  $S_i^* = S_j^*$  or  $i > j$ .

Any order of the facets satisfying (i) and (ii) is called a *shelling* of  $\Gamma$ .

In [5, Theorem 10.5] the following has been proved.

**Theorem 1.5.** *The multicomplex  $\Gamma$  is shellable if and only if  $S/I(\Gamma)$  is a pretty clean  $S$ -module.*

**Remark 1.6.** Let  $\Gamma \subset \mathbb{N}_\infty^n$  be a shellable multicomplex with shelling  $a_1, \dots, a_r$ , then  $a_1(i) \in \{0, \infty\}$  and therefore  $a_1$  is one of the minimal elements in  $F(\Gamma)$  with respect to its partially order. Indeed, since  $a_1, \dots, a_r$  is a shelling, it follows that  $S_1 = \Gamma(a_1)$  is a Stanley set and therefore there exists a vector  $b \in \mathbb{N}^n$  and a vector  $m \in \{0, \infty\}^n$  such that

$$\Gamma(a_1) = b + \Gamma(m).$$

It is clear that  $\text{infpt}(a_1) = \text{infpt}(m)$ . If  $\text{infpt}(m) = [n]$ , then there is nothing to show. Suppose now that  $\text{infpt}(m) \neq [n]$ , and choose  $i \in [n] \setminus \text{infpt}(m)$ . If  $a_1(i) \neq 0$  there exists  $c \in \Gamma(a_1)$  with  $c(i) < a_1(i)$ . Since  $c$  and  $a_1 \in b + \Gamma(m) = \Gamma(a_1)$ , and since  $m(i) = 0$ , it follows that  $c(i) = b(i) = a_1(i)$ , a contradiction.

Furthermore, if  $\Gamma$  has only one maximal facet, then  $F(\Gamma)$  has only one minimal element, and any shelling of  $\Gamma$  must start with this minimal element and end by the maximal one. In fact, suppose  $a_1$  and  $a_2$  are minimal elements in  $F(\Gamma)$ . By the first part of this remark it follows that  $a_1$  and  $a_2$  are vectors in  $\{0, \infty\}^n$ . Hence since  $\text{infpt}(a_1) = \text{infpt}(a_2)$ , we see that  $a_1 = a_2$ . Now let  $a_1, \dots, a_r$  be any shelling of  $\Gamma$ . Then, by what we have shown, it

follows that  $a_1$  is the unique minimal element in  $F(\Gamma)$ . Let  $m$  be the maximal element of  $F(\Gamma)$ . Suppose  $m = a_k$  for some  $k < r$ , then

$$S_{k+1} = \Gamma(a_{k+1}) \setminus \Gamma(a_1, \dots, a_k) = \Gamma(a_{k+1}) \setminus \Gamma(m) = \emptyset,$$

which is not a Stanley set, a contradiction. Moreover in this case for each  $i$  there exists a  $d_i \in \mathbb{N}^n$  such that  $S_i = d_i + \Gamma(a_1)$ .

Now we are ready to show that in  $S = K[x_1, \dots, x_n]$ , any ideal of height  $n - 1$  is pretty clean.

**Proposition 1.7.** *Let  $I \subset S = K[x_1, \dots, x_n]$  be any monomial ideal of height  $\geq n - 1$ . Then  $I$  is pretty clean.*

*Proof.* We may assume that  $I$  is a monomial ideal of height  $n - 1$ , and by Lemma 1.1 that  $I$  is saturated, i.e.,  $I = \tilde{I}$ . It follows that  $I = \bigcap I_j$ , where  $I_j = (x_1^{c_{j1}}, \dots, x_{j-1}^{c_{j,j-1}}, x_{j+1}^{c_{j,j+1}}, \dots, x_n^{c_{jn}})$ , and where  $c_{jk} > 0$  for  $k \neq j$ . We denote by  $\Gamma$  and  $\Gamma_j$  the multicomplexes associated to  $I$  and  $I_j$ , and by  $F$  and  $F_j$  the sets of facets of  $\Gamma$  and  $\Gamma_j$ , respectively. The sets  $F$  and  $F_j$  are finite, see [5, Lemma 9.6]. Suppose  $|F| = t$  and  $|F_j| = t_j$ . Since  $I_j$  is  $P_j$ -primary where  $P_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ , it follows from [5, Proposition 5.1] that  $I_j$  is pretty clean, and hence  $\Gamma_j$  is shellable. Moreover  $a \in \mathbb{N}_\infty^n$  is a facet of  $\Gamma_j$  if and only if  $a(j) = \infty$  and  $a(k) < c_{jk}$  for  $k \neq j$ . Let  $a_{j1}, \dots, a_{jt_j}$  be a shelling of  $\Gamma_j$ .

For showing  $I$  is pretty clean it is enough to show that  $\Gamma$  is shellable.

By [5, Lemma 9.9 (b)] we have  $\Gamma = \bigcup_{j=1}^n \Gamma_j$ . Also by [5, Lemma 9.10], each  $F_j$  has only one maximal facet, say  $m_j$ , where

$$m_j(k) = \begin{cases} \infty, & \text{if } k = j, \\ c_{jk} - 1, & \text{otherwise.} \end{cases}$$

It follows that  $F = \bigcup F_j$  and that the union is disjoint, since  $a \in F$  belongs to  $F_j$  if and only if  $a(j) = \infty$  and  $a(k) < \infty$  for  $k \neq j$ . In particular one has  $(\bigcup_{i=1}^{j-1} F_i) \cap F_j = \emptyset$  for  $j = 2, \dots, n$ .

We claim that

$$a_{11}, \dots, a_{1t_1}, a_{21}, \dots, a_{2t_2}, \dots, a_{n1}, \dots, a_{nt_n}$$

is a shelling for  $\Gamma$ . Indeed, for all  $j$  and all  $k$  with  $1 < k \leq t_j$  we have

$$S_{jk} = \Gamma(a_{jk}) \setminus \Gamma(a_{11}, \dots, a_{jk-1}) = \Gamma(a_{jk}) \setminus \Gamma(a_{j1}, \dots, a_{jk-1}),$$

and if  $k = 1$ , then

$$S_{j1} = \Gamma(a_{j1}) \setminus \Gamma(a_{11}, \dots, a_{j-1t_{j-1}}) = \Gamma(a_{j1}).$$

Since  $a_{j1}, \dots, a_{jt_j}$  is a shelling of  $\Gamma_j$ , it follows that  $S_{jk}$  is a Stanley set for all  $j$  and all  $k$ .

Condition (ii) in the definition of shellability is obviously satisfied. In fact, since  $\Gamma_j$  is shellable and has only one maximal facet, it follows by Remark 1.6 that for all  $k = 1, \dots, t_j$ , there exists some  $d_{jk} \in \mathbb{N}^n$  such that  $S_{jk} = d_{jk} + S_j^*$ , where  $S_j^* = \Gamma(a_{j1})$ . Moreover if  $j \neq t$  then  $a_{j1}$  and  $a_{t1}$  are not comparable, and hence in this case there is no inclusion among  $S_j^*$  and  $S_t^*$ .  $\square$

As a Consequence of Proposition 1.7 we have

**Corollary 1.8.** *Any monomial ideal  $I \subset S = K[x, y]$  is pretty clean.*

Next we will show that any monomial ideal in  $S = K[x_1, x_2, x_3]$  is also pretty clean. First we need

**Lemma 1.9.** *Let  $I \subset S = K[x_1, x_2, x_3]$  be a monomial ideal of height 1. Then  $I = uJ$ , where  $u$  is a monomial in  $S$ , and  $J$  is a monomial ideal of height  $\geq 2$ . Moreover,  $I$  is pretty clean if and only if  $J$  is pretty clean.*

*Proof.* The first statement of the lemma is obvious. Assume now that  $J$  is pretty clean with pretty clean filtration

$$\mathcal{F} : J = J_0 \subset J_1 \subset \dots \subset J_r = S$$

such that  $J_i/J_{i-1} \cong S/P_i$ , where  $P_i \in \text{Ass} J$ . Then  $\text{height } P_i \geq 2$ . It follows that

$$\mathcal{F}_1 : I = uJ \subset uJ_1 \subset \dots \subset uJ_r = (u)$$

is a prime filtration of  $(u)/I$  with factors  $uJ_i/uJ_{i-1} \cong S/P_i$ .

There exists a prime filtration

$$\mathcal{F}_2 : (u) = J_r \subset J_{r+1} \subset \dots \subset J_{r+t} = S$$

of the principal monomial ideal  $I_1 = (u)$ , where the  $J_{r+k}$  are again principal monomial ideals with  $J_{r+k}/J_{r+k-1} \cong S/Q_k$  and where  $Q_k \in \text{Ass}(u)$  has height 1 for all  $k$ . In fact, if  $u = u_0 = \prod_{i=1}^k x_{i_1}^{a_i}$  and  $u_j = \prod_{r=j+1}^k x_{i_r}^{a_r}$  for  $j = 1, \dots, k-1$ , then the prime filtration  $\mathcal{F}_2$  is the following:

$$\mathcal{F}_2 : J_r = (u) \subset (x_{i_1}^{a_1-1} u_1) \subset (x_{i_1}^{a_1-2} u_1) \dots \subset (u_1) \subset (x_{i_2}^{a_2-1} u_2) \subset \dots \subset (u_2) \subset \dots \subset (x_{i_k}) \subset S.$$

Therefore this filtration of  $I_1 = (u)$  is pretty clean. Now composing the above filtrations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  we obtain a pretty clean filtration of  $I$ .

The converse follows from Proposition 1.7, because  $\text{height}(J) \geq 2$ .  $\square$

Combining Lemma 1.9 with Proposition 1.7 we get

**Theorem 1.10.** *Any monomial ideal in a polynomial ring in at most three variables is pretty clean.*

The following example shows that this theorem can not be extended to polynomial rings in more than three variables, and it also shows that monomial ideals of height  $< n-1$  may not be pretty clean.

**Example 1.11.** Let  $n = 4$ , and  $\Gamma$  be the multicomplex with facets  $(\infty, \infty, 0, 0)$  and  $(0, 0, \infty, \infty)$ . Then  $\Gamma$  is not shellable, and so the monomial ideal

$$I(\Gamma) = (x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4) \subset K[x_1, x_2, x_3, x_4]$$

is not pretty clean.

More generally, let  $n > 3$  and  $a = (0, 0, \infty, \dots, \infty)$  and  $b = (\infty, \infty, 0, \dots, 0)$  be two elements in  $\mathbb{N}_\infty^n$ . Then  $\Gamma = \Gamma(a, b)$  is not a shellable multicomplex, hence  $I = (x_1, x_2) \cap (x_3, \dots, x_n)$  is a square free monomial ideal in  $S = K[x_1, \dots, x_n]$  which is not clean.



## 2. PRIME FILTRATIONS AND STANLEY DECOMPOSITIONS

Let  $I \subset S = K[x_1, \dots, x_n]$  be a monomial ideal, any decomposition of  $S/I$  as a direct sum of  $K$ -vectorspaces of the form  $uK[Z]$  where  $u$  is a monomial in  $K[X]$ , and  $Z \subset X = \{x_1, \dots, x_n\}$  is called a *Stanley decomposition*. In this paper we will call  $uK[Z]$  a Stanley space of dimension  $|Z|$ , where  $|Z|$  denotes the cardinality of  $Z$ . Stanley decomposition have been studied in various combinatorial and algebraic contexts, see [1],[6], and [7].

Let  $R$  be a finitely generated standard graded  $K$ -algebra where  $K$  is a field, and let  $M$  be a finitely generated graded  $R$ -module. Then the Hilbert series of  $M$  is defined to be  $\text{Hilb}(M) = \sum_{i \in \mathbb{Z}} \dim_K M_i t^i$ . It is known that if  $\dim(M) = d$ , then there exists a  $Q_M(t) \in \mathbb{Z}[t, t^{-1}]$  such that

$$\text{Hilb}(M) = Q_M(t)/(1-t)^d$$

and  $Q_M(1) \neq 0$ . The number  $Q_M(1)$  is called the multiplicity of  $M$ , and is denoted by  $e(M)$ .

Let  $I \subset S$  be a monomial ideal. Then the number of Stanley spaces of a given dimension in a Stanley decomposition may depend on this particular decomposition. For example, if  $I = (xy) \subset K[x, y]$ , then for all integers  $k > 0$  and  $l > 0$  we have the Stanley decomposition

$$S/I = x^l K[x] \oplus y^k K[y] \oplus \left( \bigoplus_{i=0}^{l-1} x^i K \right) \oplus \left( \bigoplus_{j=1}^{k-1} y^j K \right),$$

for  $S/I$  with as many Stanley spaces of dimension 0 as we want, however only 2 Stanley spaces of dimension 1 in any Stanley decomposition.

This is a general fact. Indeed, the number of Stanley spaces of maximal dimension is independent of the special Stanley decomposition. In fact, this number is equal to the multiplicity,  $e(S/I)$ , of  $S/I$ .

Let

$$S/I = \bigoplus_{i=1}^r u_i K[Z_i]$$

be an arbitrary Stanley decomposition of  $S/I$ , and  $d = \max\{|Z_i| : i = 1, \dots, r\}$ . Then

$$\text{Hilb}(S/I) = \sum_{i=1}^r \text{Hilb}(u_i K[Z_i]) = \sum_{i=1}^r t^{\deg(u_i)} / (1-t)^{|Z_i|} = Q_{S/I}(t)/(1-t)^d.$$

with  $Q_{S/I}(t) = \sum_{i=1}^r (1-t)^{d-|Z_i|} t^{\deg(u_i)}$ . It follows that  $e(S/I) = Q_{S/I}(1)$  is equal to the number of Stanley space of dimension  $d$  in this Stanley decomposition of  $S/I$ .

We also note that for each monomial  $u \in \tilde{I} \setminus I$  the 0-dimensional Stanley space  $uK$  belongs to any Stanley decomposition of  $S/I$ . In fact  $um^k \subset I$  for some  $k$ . Now if  $u$  belongs to some Stanley space  $vK[Z]$  with  $|Z| \geq 1$ , then  $vK[Z] \cap I \neq \emptyset$ , a contradiction.

Stanley [12] conjectured that there always exists a Stanley decomposition

$$S/I = \bigoplus_{i=1}^r u_i K[Z_i],$$

such that  $|Z_i| \geq \text{depth}(S/I)$  for all  $i$ .

Apel [1] studied some cases in which Stanley's conjecture holds. Also in [5, Theorem 6.5] it has been proved that for all pretty clean monomial ideals Stanley's conjecture holds. Therefore combining Theorem 1.10 and Lemma 1.7 with [5, Theorem 6.5] we get

**Proposition 2.1.** (a) *Let  $I \subset S = K[x_1, \dots, x_n]$  be a monomial ideal of height  $\geq n - 1$ . Then Stanley's conjecture holds for  $S/I$ .*

(b) (Apel, [1, Theorem 5.1]) *Let  $I$  be any monomial ideal in the polynomial ring in at most three variables. Then Stanley's conjecture holds for  $S/I$ .*

In the proof of [5, Theorem 6.5] it is used that Stanley decompositions of  $S/I$  arise from prime filtrations. In fact, if  $\mathcal{F}$  is a prime filtration of  $S/I$  with factors  $(S/P_i)(-a_i)$ ,  $i = 1, \dots, r$ . Then if we set  $u_i = \prod_{j=1}^n x_j^{a_i(j)}$  and  $Z_i = \{x_j : x_j \notin P_i\}$ , then

$$S/I = \bigoplus_{i=1}^r u_i K[Z_i].$$

Recall that if  $\mathcal{F}$  is a pretty clean filtration of  $S/I$ , then  $\text{Ass}(S/I) = \text{Supp}(\mathcal{F})$ . The converse of this statement is not always true, see Example 4.4. As a generalization of [5, Theorem 6.5] we show

**Proposition 2.2.** *Suppose  $I \subset S$  is a monomial ideal, and  $\mathcal{F}$  is a prime filtration of  $S/I$  with  $\text{Supp}(\mathcal{F}) = \text{Ass}(S/I)$ . Then the Stanley decomposition of  $S/I$  which is obtained from this prime filtration satisfies the condition of Stanley's conjecture.*

*Proof.* The Stanley decomposition which is obtained from  $\mathcal{F}$  has the property that  $|Z_i| = \dim S/P_i$ . By [2, Proposition 1.2.13] we have  $\text{depth}(S/I) \leq \dim(S/P_i)$  for all  $P_i \in \text{Ass}(S/I)$ , and hence the assertion follows.  $\square$

In all cases discussed above we found a Stanley decomposition corresponding to a prime filtration and satisfying the Stanley conjecture. However we will show that there exist examples of monomial ideals such that *all* Stanley decompositions arising from a prime filtration may fail to satisfy the Stanley conjecture.

First we notice that

**Remark 2.3.** Let  $I \subset S$  be a Cohen-Macaulay monomial ideal, and

$$\mathcal{F} : I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

be a prime filtration of  $S/I$ . We claim that if the Stanley decomposition of  $S/I$  corresponding to  $\mathcal{F}$  satisfies the Stanley conjecture, then  $\text{Ass}(I) = \text{Supp}(\mathcal{F})$ . In particular  $I$  is clean, since  $\text{Min}(I) = \text{Ass}(I)$ .

Indeed, since  $I$  is Cohen-Macaulay we have  $\text{depth}(S/I) = \dim(S/I) = \dim(S/P)$  for all  $P \in \text{Ass}(I)$ . We recall that  $I_i/I_{i-1} \cong S/P_i(-a_i)$  for suitable  $a_i$  and that  $P_i \in \text{Ass}(I_{i-1})$  for  $i = 1, \dots, r$ . Let  $T_i = u_i K[Z_i]$  be the Stanley space corresponding to  $S/P_i(-a_i)$  as explained as above. Then  $|Z_i| = \dim(S/P_i)$ . Assume that  $P_i \notin \text{Ass}(I)$  for some  $i > 1$ . Since  $I \subset I_{i-1} \subset P_i$ , there exists a  $P_j \in \text{Ass}(I)$  such that  $P_j \subsetneq P_i$ . It follows that  $|Z_i| = \dim(S/P_i) < \dim(S/P_j) = \text{depth}(S/I)$ , a contradiction.

**Example 2.4.** Let  $K$  be a field and

$$I = (abd, abf, ace, adc, aef, bde, bcf, bce, cdf, def) \subset S = K[a, b, c, d, e, f].$$



The ideal  $I$  is the Stanley-Reisner ideal corresponding to the simplicial complex  $\Delta$  which is the triangulation of the real projective plane  $\mathbb{P}^2$ , see [2, Figure 5.8]. It is known that  $S/I$  is Cohen-Macaulay if and only if  $\text{char}(K) \neq 2$ . This implies  $S/I$  is not clean, since otherwise  $\Delta$  would be shellable and  $S/I$  would be Cohen-Macaulay for any field  $K$ . Hence by Remark 2.3, if  $\text{char}(K) \neq 2$ , no Stanley decomposition of  $S/I$  which corresponds to a prime filtration of  $S/I$  satisfies the Stanley conjecture.

Unfortunately not all Stanley decompositions of  $S/I$  correspond to prime filtrations, even if  $S/I$  is pretty clean. Such an example is given by McLagan and Smith in [7]. Let  $I = (x_1x_2x_3) \subset K[x_1, x_2, x_3]$ . Then

$$S/I = 1 \oplus x_1K[x_1, x_2] \oplus x_2K[x_2, x_3] \oplus x_3K[x_1, x_3]$$

is a Stanley decomposition of  $S/I$  which does not correspond to a prime filtration of  $S/I$ . On the other hand, by Theorem 1.10 we know that  $S/I$  is a pretty clean.

Now we want to characterize those Stanley decompositions of  $S/I$  which correspond to a prime filtration of  $S/I$ .

We first notice

**Lemma 2.5.** *Let  $I \subset S = K[x_1, \dots, x_n]$  be a monomial ideal, and  $T = uK[Z]$  be a Stanley space in a Stanley decomposition of  $S/I$ . The  $K$ -vectorspace  $I_1 = I \oplus T$  is a monomial ideal if and only if  $I_1 = (I, u)$ . In this case,  $I : u = P$ , where  $P = (x_i : x_i \notin Z)$ .*

*Proof.* We have  $I \subset I_1$  and  $u \in I_1$ . Suppose now that  $I_1$  is a monomial ideal. Since  $(I, u)$  is the smallest monomial ideal that contains  $I$  and  $u$ , it follows that  $(I, u) \subset I_1$ . On the other hand,  $I_1 = I + uK[Z] \subset I + uK[x_1, \dots, x_n] = (I, u)$ . Hence  $I_1 = (I, u)$ .

Since for each  $x_i \notin Z$  we have  $x_iu \in I_1 = I \oplus T$  and  $x_iu \notin uK[Z] = T$ , it follows that  $x_iu \in I$  and hence  $x_i \in I : u$ . On the other hand, if  $v \in K[Z]$  is a monomial, then  $vu \notin I$ , since  $uK[Z]$  is a Stanley space of  $S/I$ . Therefore  $I : u = P = (x_i : x_i \notin Z)$ .  $\square$

**Corollary 2.6.** *The monomial ideal  $I \subset S$  is a prime ideal if and only if there exists a Stanley decomposition of  $S/I$  consisting of only one Stanley space.*

As a consequence of this Lemma we have

**Proposition 2.7.** *Let  $I \subset S$  be a monomial ideal, and  $S/I = \bigoplus_{i=1}^r u_iK[Z_i]$  be a Stanley decomposition of  $S/I$ . The given Stanley decomposition corresponds to a prime filtration of  $S/I$  if and only if the Stanley spaces  $T_i = u_iK[Z_i]$  can be ordered  $T_1, \dots, T_r$ , such that*

$$I_k = I \oplus T_1 \oplus \dots \oplus T_k$$

*is a monomial ideal for  $k = 1, \dots, r$ .*

*Proof.* We prove "if" by induction on  $r$ . If  $r = 0$  then the assertion is trivially true. Let  $r \geq 1$ . By assumption  $I_1 = I \oplus T_1$  is a monomial ideal. Hence by Lemma 2.5 we have  $I_1 = (I, u_1)$  and  $I : u_1 = P_1 = (x_i : x_i \notin Z_1)$ . We notice that in this case  $I_1/I \cong S/P_1(-a_1)$  and  $u_1 = \prod_{j=1}^n x_j^{a_1(j)}$ , and that  $S/I_1 = \bigoplus_{i=2}^r T_i$ . Now by the induction hypothesis this Stanley decomposition of  $S/I_1$  corresponds to a prime filtration, say  $\mathcal{F}_1$

$$\mathcal{F}_1 : I_1 \subset I_2 \subset \dots \subset I_r = S.$$

Therefore the given Stanley decomposition of  $S/I$  corresponds to the prime filtration

$$\mathcal{F} : I \subset I_1 \subset I_2 \subset \dots \subset I_r = S.$$

The converse follows immediately if we order the Stanley spaces of  $S/I$  which are obtained from a prime filtration according to the order of the ideals in this filtration.  $\square$

We conclude this section by showing

**Corollary 2.8.** *Let  $I \subset S = K[x, y]$  be a monomial ideal. Then each Stanley decomposition of  $S/I$  corresponds to a prime filtration of  $S/I$ .*

*Proof.* The  $K$ -vector-space  $\tilde{I}/I$  has finite dimension, say  $m$ . So we can choose monomials  $v_1, \dots, v_m \in \tilde{I}$  whose residue classes modulo  $I$  form a  $K$ -basis for  $\tilde{I}/I$ . As observed in the discussions before Proposition 2.1, in any Stanley decomposition of  $S/I$  these monomials have to appear as 0-dimensional Stanley spaces. In the proof of Lemma 1.1 we showed that it is possible to order the monomials  $v_1, \dots, v_m$  in such a way that

$$I_i = I \oplus v_1 K \oplus \dots \oplus v_i K = (I, v_1, \dots, v_i)$$

is a monomial ideal for  $i = 1, \dots, m$ . If we remove in the given Stanley decomposition of  $S/I$  the Stanley spaces  $v_i K$ ,  $i = 1, \dots, m$ , the remaining summands establish a Stanley decomposition of  $S/\tilde{I}$ . Thus we may assume that  $I$  is saturated. Hence  $I = (x^\alpha y^\beta)$ .

Let  $S/I = \bigoplus_{i=1}^r u_i K[Z_i]$  be a Stanley decomposition of  $S/I$ . We will prove by induction on  $\alpha + \beta$  that the given Stanley decomposition can be ordered such that  $I_k = I \oplus (\bigoplus_{i=1}^k u_i K[Z_i])$  is a monomial ideal for all  $k$ . If  $\alpha + \beta = 0$  the assertion is trivially true. Let  $\alpha + \beta > 0$ . The Stanley decomposition of  $S/I$  contains at least one summand of the form  $x^{\alpha-1} y^\gamma K[y]$ , where  $\gamma \geq \beta$ , or  $x^\theta y^{\beta-1} K[x]$ , where  $\theta \geq \alpha$ .

We may assume that  $x^{\alpha-1} y^\gamma K[y]$  is one of the summands. Let  $t = \gamma - \beta$ , and set  $v_i = x^{\alpha-1} y^{\gamma-i+1}$  for  $i = 1, \dots, t+1$ . If we set  $T_1 = v_1 K[y]$ , then  $I_1 = I \oplus T_1 = (I, v_1)$  is a monomial ideal. If we remove the Stanley space  $T_1$  from the given Stanley decomposition of  $S/I$ , the remaining establish a Stanley decomposition of  $S/I_1$ . Since  $v_2, \dots, v_{t+1}$  belong to  $\tilde{I}_1 \setminus I_1$ , these monomials have to appear in any Stanley decomposition of  $S/I_1$  as 0-dimensional Stanley spaces. In particular these monomials appear as 0-dimensional Stanley space,  $T_2 = v_2 K, \dots, T_{t+1} = v_{t+1} K$  in the given Stanley decomposition of  $S/I$ . Now it is clear that  $I_i = I_{i-1} \oplus T_i = (I_{i-1}, v_i)$  is a monomial ideal for  $i = 1, \dots, t+1$ , where  $I_0 = I$ .

Removing the Stanley spaces  $T_1, \dots, T_{t+1}$  from the given Stanley decomposition of  $S/I$ , the remaining summands establish a Stanley decomposition of  $S/I_{t+1}$ . Since  $I_{t+1} = (x^{\alpha-1} y^\beta)$  is a saturated ideal, the assertion follows by the induction hypothesis applied to  $S/I_{t+1}$ .  $\square$

### 3. A CHARACTERIZATION OF PRETTY CLEAN MONOMIAL IDEALS IN TERMS OF POLARIZATIONS

In this section we consider polarizations of monomial ideals and of prime filtrations. Let  $S = K[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over the field  $K$ , and  $u =$

$\prod_{i=1}^n x_i^{a_i}$  be a monomial in  $S$ . Then

$$u^p = \prod_{i=1}^n \prod_{j=1}^{a_i} x_{ij} \in K[x_{11}, \dots, x_{1a_1}, \dots, x_{n1}, \dots, x_{na_n}]$$

is called the *polarization* of  $u$ .

Let  $I$  be a monomial ideal in  $S$  with monomial generators  $u_1, \dots, u_m$ . Then  $(u_1^p, \dots, u_m^p)$  is called a *polarization* of  $I$ . Note that if  $v_1, \dots, v_k$  is another set of monomial generators of  $I$  and if  $T$  is the polynomial with sufficiently many variables  $x_{ij}$  such that all the monomials  $u_i^p$  and  $v_j^p$  belong to  $T$ , then

$$(u_1^p, \dots, u_m^p)T = (v_1^p, \dots, v_k^p)T.$$

Therefore we denote any polarization of  $I$  by  $I^p$ , since in a common polynomial ring extension all polarizations are the same, and we write  $I^p = J^p$  if a polarization of  $I$  and a polarization of  $J$  coincide in a common polynomial ring extension.

Now let  $I = (u_1, \dots, u_m) \subset S$  be a monomial ideal, and  $u \in S$  a monomial. Furthermore let  $T$  be the polynomial ring in variables  $x_{ij}$  such that:

- (1) for all  $i \in [n]$  there exists  $k_i \geq 1$  such that  $x_{i1}, \dots, x_{ik_i}$  are in  $T$ ,
- (2)  $I^p \subset T$ , and  $u^p \in T$ .

We consider the  $K$ -algebra homomorphism

$$\pi : T \longrightarrow S, \quad x_{ij} \mapsto x_i.$$

Then  $\pi$  is an epimorphism with  $\pi(u^p) = u$  for all monomials  $u \in S$ , and  $u^p$  is the unique squarefree monomial in  $T$  of the form  $\prod_{i=1}^n \prod_{j=1}^{t_i} x_{ij}$  with this property. In particular,  $\pi(I^p) = I$ . We call  $\pi$  the specialization map attached with the polarization.

**Remark 3.1.** Let  $I = (u_1, \dots, u_m) \subset S$  be a monomial ideal, and  $u \in S$  a monomial. Then

- (a)  $I : u = (u_i / \gcd(u_i, u))_{i=1}^m$ , and it is again a monomial ideal in  $S$ .
- (b)  $I : u$  is a prime ideal if and only if for each  $i \in [m]$ , there exists a  $j \in [n]$  such that  $u_j / \gcd(u_j, u)$  is a monomial of degree one, and  $u_j / \gcd(u_j, u)$  divides  $u_i / \gcd(u_i, u)$ .
- (c) Let  $u = \prod_{i=1}^n x_i^{a_i}$  and  $u_j = \prod_{i=1}^n x_i^{b_i}$ . If  $u_j / \gcd(u_j, u) = x_i$ , then  $b_i = a_i + 1$  and  $b_t \leq a_t$  for all  $t \neq i$ . Therefore  $u_j / \gcd(u_j, u) = x_i$  if and only if  $u_j^p / \gcd(u_j^p, u^p) = x_{ib_i}$ .

**Lemma 3.2.** Let  $I = (u_1, \dots, u_m) \subset S$  be a monomial ideal and  $u \in S$  a monomial. If  $I^p : u^p$  is a prime ideal, then  $I^p : u^p = (x_{i_1 j_1}, \dots, x_{i_k j_k})$  with  $i_r \neq i_s$  for  $r \neq s$ .

*Proof.* Since  $I^p : u^p$  is a monomial prime ideal in polynomial ring  $T$  it must be generated by variables. If  $x_{ij}$  and  $x_{ik}$  are two generators of  $I^p : u^p$ , then there exist  $r_j \in [m]$ , and  $r_k \in [m]$  such that  $x_{ij} = u_{r_j}^p / \gcd(u_{r_j}^p, u^p)$  and  $x_{ik} = u_{r_k}^p / \gcd(u_{r_k}^p, u^p)$ . It follows from Remark 3.1(c) that  $j - 1 = k - 1$  is equal to the exponent of  $x_i$  in  $u$ . Hence  $x_{ij} = x_{ik}$ .  $\square$

We also need to show

**Lemma 3.3.** Let  $I = (u_1, \dots, u_m) \subset S$  be a monomial ideal, and  $u \in S$  a monomial in  $S$ . Then  $I : u$  is a prime ideal if and only if  $I^p : u^p$  is a prime ideal. In this case  $I : u = \pi(I^p : u^p)$ .

*Proof.* Let  $I : u$  be a prime ideal. We may assume that  $I : u = (x_1, \dots, x_k)$  for some  $k \in [n]$ . Therefore for each  $i \in [k]$  there exists some  $u_{j_i}$ , with  $j_i \in [m]$ , such that  $x_i = u_{j_i} / \gcd(u_{j_i}, u)$  and for each  $t \in [m]$ , there exists  $i_t \in [k]$ , such that  $x_{i_t}$  divides  $(u_t / \gcd(u_t, u))$ . Therefore by Remark 3.1(c) we have  $u_{j_i}^p / \gcd(u_{j_i}^p, u^p) = x_{i_t i}$ , where  $t_i$  is the exponent of  $x_i$  in  $u_{j_i}$  and  $t_i - 1$  is the exponent of  $x_i$  in  $u$ .

Also for each  $s \in [m]$ , the monomial  $u_s^p / \gcd(u_s^p, u^p)$  is divided by one of these  $x_{i_t i}$ , where  $i \in [k]$ . Indeed, since  $I : u$  is a prime ideal there exists some  $i \in [k]$  such that  $x_i$  divides  $(u_s / \gcd(u_s, u))$ , where  $x_i = u_{j_i} / \gcd(u_{j_i}, u)$ . Let  $t_i - 1$  be the exponent of  $x_i$  in  $u$ . Then it follows that the exponent of  $x_i$  in  $u_s$  is  $> t_i - 1$ . Hence  $x_{i_t i}$  divides  $u_s^p / \gcd(u_s^p, u^p)$ , and  $I^p : u^p = (x_{1t_1}, \dots, x_{kt_k})$ .

For the converse, let  $I^p : u^p$  be a prime ideal. By Lemma 3.2 we may assume that  $I^p : u^p = (x_{1t_1}, \dots, x_{kt_k})$ . This means that for each  $i \in [k]$  there is a monomial  $u_{j_i}$  with  $j_i \in [m]$  such that  $x_{i_t i} = u_{j_i}^p / \gcd(u_{j_i}^p, u^p)$  and for each  $s \in [m]$ , the squarefree monomial  $u_s^p / \gcd(u_s^p, u^p)$  is divided by one of these  $x_{i_t i}$ . Therefore by Remark 3.1(c) we have  $x_i = u_{j_i} / \gcd(u_{j_i}, u)$  for  $i \in [k]$ , and for each  $s \in [m]$ , one of these variables divides  $u_s / \gcd(u_s, u)$ . Hence  $I : u = (x_1, \dots, x_k)$ .  $\square$

Let  $I \subset S$  be a monomial ideal and

$$\mathcal{F} : I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

a filtration of  $S/I$ . We call  $r$  the *length of filtration*  $\mathcal{F}$  and denote it by  $\ell(\mathcal{F})$ .

Assume now that for all  $j$  we have  $I_{j+1} = (I_j, u_j)$  where  $u_j \in S$  is a monomial. We will define the *polarization*  $\mathcal{F}^p$  of  $\mathcal{F}$  inductively as follow: set  $J_0 = I^p$ ; assuming that  $J_i$  is already defined, we set  $J_{i+1} = (J_i, u_i^p)$ . So  $J_i = (I^p, u_1^p, \dots, u_i^p)$ , and

$$\mathcal{F}^p : I^p = J_0 \subset J_1 \subset \dots \subset J_r = T$$

is a filtration of  $T/I^p$ .

We have the following

**Proposition 3.4.** *Suppose  $I \subset S$  is a monomial ideal, and*

$$\mathcal{F} : I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

*a filtration of  $S/I$  as above. Then  $\mathcal{F}$  is a prime filtration of  $S/I$  if and only if  $\mathcal{F}^p$  is a prime filtration of  $T/I^p$ .*

*Proof.* Let

$$\mathcal{F} : I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

be a prime filtration of  $S/I$ . We use induction on  $r = \ell(\mathcal{F})$  the length of prime filtration. If  $r = 1$ , then  $I$  is a monomial prime ideal and  $I^p = I$ .

Let  $r > 1$ . Then  $\mathcal{F}_1 : I_1 \subset \dots \subset I_r = S$  is a prime filtration of  $S/I_1$ , and  $\ell(\mathcal{F}_1) = r - 1$ . By our induction hypothesis,  $\mathcal{F}_1^p$  is a prime filtration of  $I_1^p = (I^p, u_1^p)$ . Since  $I_1/I \cong I_1 : u_1$  is a prime ideal, it follows from Lemma 3.3 that  $J_0/J_1 \cong I_1^p : u_1^p$  is a prime ideal too. Hence  $\mathcal{F}^p$  is a prime filtration of  $T/I^p$ .

The other direction of the statement is proved similarly.  $\square$

Let  $S = K[x_1, \dots, x_n]$  be the polynomial ring, and  $u, v \in S$  be monomials. We notice that

$$\text{lcm}(u, v)^p = \text{lcm}(u^p, v^p).$$

Therefore we have

**Lemma 3.5.** *Let  $I, J$  be two monomial ideals in  $S$ . Then  $(I \cap J)^p = I^p \cap J^p$ .*

*Proof.* Let  $I = (u_1, \dots, u_m)$  and  $J = (v_1, \dots, v_t)$ . Then  $I \cap J = (\text{lcm}(u_i, v_j))$ , where  $1 \leq i \leq m$  and  $1 \leq j \leq t$ . Therefore  $(I \cap J)^p = (\text{lcm}(u_i, v_j)^p) = (\text{lcm}(u_i^p, v_j^p)) = I^p \cap J^p$ .  $\square$

We recall that a monomial ideal  $I \subset S$  is an irreducible monomial ideal if and only if there exists a subset  $A \subset [n]$  and for each  $i \in A$  an integer  $a_i > 0$  such that  $I = (x_i^{a_i} : i \in A)$ , see [14, Theorem 5.1.16]. It is known that for each minimal ideal  $I$  there exists a decomposition  $I = \bigcap_{i=1}^r J_i$  such that  $J_i$  are irreducible monomial ideals.

**Corollary 3.6.** *Suppose  $J_1, \dots, J_r$  are monomial ideals in the polynomial ring  $S$ , and  $I = \bigcap_{i=1}^r J_i$ . Then  $I^p = \bigcap_{i=1}^r J_i^p$ . In particular the minimal prime ideals of  $I^p$  are of the form  $(x_{i_1 t_1}, \dots, x_{i_k t_k})$ , with  $i_r \neq i_s$  for  $r \neq s$ .*

Next we show that if  $I \subset S$  is a monomial ideal and  $I^p$  the polarization of  $I$ , then  $|F(\Gamma(I))| = |F(\Gamma(I^p))|$ . First we notice the following:

**Lemma 3.7.** *Let  $I \subset S$  be an irreducible monomial ideal and  $I^p$  the polarization of  $I$ . Furthermore, let  $F$  and  $F^p$  be the sets of facets of  $\Gamma(I)$  and  $\Gamma(I^p)$ , respectively. Then there exists a bijection between  $F$  and  $F^p$ .*

*Proof.* By [14, Theorem 5.1.16] there exists a subset  $A \subset [n]$  and for each  $i \in A$  an integer  $a_i > 0$  such that  $I = (x_i^{a_i} : i \in A)$ . We may assume  $A = [k]$  for some  $k \leq n$ . In this case  $\Gamma(I) = \Gamma(m)$ , where

$$m(i) = \begin{cases} a_i - 1, & \text{if } i \in [k], \\ \infty, & \text{otherwise,} \end{cases}$$

and  $a \in F$  if and only if  $a \leq m$  and  $a(i) = \infty$  for  $i > k$ . We have

$$I^p = \left( \prod_{j=1}^{a_1} x_{1j}, \prod_{j=1}^{a_2} x_{2j}, \dots, \prod_{j=1}^{a_k} x_{kj} \right),$$

and we know that the facets in  $F^p$  correspond to the minimal prime ideals of  $I^p$ . Indeed, if  $a \in F^p$  is a facet of  $\Gamma^p$ , then  $P_a = (x_i : a(i) = 0)$  is a minimal prime ideal of  $I^p$ . Each minimal prime ideal of  $I^p$  is of the form  $(x_{1t_1}, \dots, x_{kt_k})$ , with  $t_i \leq a_i$ .

Now we define

$$\theta : F \rightarrow F^p, a \mapsto \bar{a}$$

as follows: if  $k < i \leq n$ , then  $\bar{a}(ij) = \infty$  for all  $j$ , and if  $i \in [k]$  we have  $a(i) = t_i < a_i$ , and we set

$$\bar{a}(ij) = \begin{cases} 0, & \text{if } j = t_i + 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Obviously  $\bar{a} \in F^p$ , since  $P_{\bar{a}} = (x_{1t_1+1}, \dots, x_{kt_k+1})$  is a minimal prime ideal of  $I^p$ , and it is also clear that  $\theta$  is an injective map.

Let  $\bar{a} \in F^p$ . Then  $\bar{a}$  corresponds to the minimal prime ideal  $P_{\bar{a}} = (x_{1t_1}, \dots, x_{kt_k})$ , where  $t_i \leq a_i$ . Therefore if  $k < i \leq n$ , we have  $\bar{a}(ij) = \infty$  for all  $j$ , and if  $i \in [k]$ , then

$$\bar{a}(ij) = \begin{cases} 0, & \text{if } j = t_i, \\ \infty, & \text{otherwise.} \end{cases}$$

Let  $a \in \mathbb{N}_{\infty}^n$  be the following:

$$a(i) = \begin{cases} t_i - 1, & \text{if } i \in [k], \\ \infty, & \text{otherwise,} \end{cases}$$

then  $a$  is a facet in  $F$ , since  $a \leq m$  and  $\text{infpt}(a) = n - k = \text{infpt}(m)$ , and moreover  $\theta(a) = \bar{a}$ .  $\square$

Now let  $I = (u_1, \dots, u_m) \subset S$  be a monomial ideal and let  $D \subset [n]$  be the set of elements  $i \in [n]$  such that  $x_i$  divides  $u_j$  for at least one  $j = 1, \dots, m$ . Then we set

$$r_i = \max\{t : x_i^t \text{ divides } u_j \text{ at least for one } j \in [m]\}$$

if  $i \in D$  and  $r_i = 1$ , otherwise. Moreover we set  $r = \sum_{i=1}^n r_i$ .

Note that  $I$  has a decomposition  $I = \bigcap_{i=1}^t J_i$  where the ideals  $J_i$  are irreducible monomial ideals. In other words, each  $J_i$  is generated by pure powers of some of the variables. Then  $I^p = \bigcap_{i=1}^t J_i^p$  is an ideal in the polynomial ring

$$T = K[x_{11} \cdots, x_{1r_1}, x_{21} \cdots, \dots, x_{n1}, \dots, x_{nr_n}]$$

in  $r$  variables.

We denote by  $\Gamma$ ,  $\Gamma^p$ ,  $\Gamma_i$  and  $\Gamma_i^p$  the multicomplexes associated to  $I$ ,  $I^p$ ,  $J_i$  and  $J_i^p$ , respectively, and by  $F$ ,  $F^p$ ,  $F_i$  and  $F_i^p$  the sets of facets of  $\Gamma$ ,  $\Gamma^p$ ,  $\Gamma_i$  and  $\Gamma_i^p$ , respectively.

It is clear that  $F \subset \bigcup_{i=1}^t F_i$  since  $\Gamma = \bigcup_{i=1}^t \Gamma_i$ , and also that  $F^p \subset \bigcup_{i=1}^t F_i^p$ . Each  $\Gamma_i$  has only one maximal facet, say  $m_i$ , and  $m_i(k) \leq r_k - 1$  if  $m_i(k) \neq \infty$ .

let  $A \subset \mathbb{N}_{\infty}^n$  be the following set:

$$A = \{a \in \mathbb{N}_{\infty}^n : a(i) < r_i \text{ if } a(i) \neq \infty\}.$$

We define the map

$$\beta : A \rightarrow \{0, \infty\}^r, a \mapsto \bar{a}$$

as follows: if  $a(i) = \infty$ , then  $\bar{a}(ij) = \infty$  for all  $j$ , and if  $a(i) = e$  where  $e \leq r_i - 1$ , then

$$\bar{a}(ij) = \begin{cases} 0, & \text{if } j = e + 1, \\ \infty, & \text{otherwise.} \end{cases}$$

**Proposition 3.8.** *With the above assumptions and notation the restriction of the map  $\beta$  to  $F$  is a bijection from  $F$  to  $F^p$ .*

*Proof.* First of all we want to show that  $\bar{a} \in F^p$ . Indeed,  $a \in F \subset \bigcup_{i=1}^t F_i$ . Therefore there exists an integer  $j \in [n]$  such that  $a \in F_j$ , and since the restriction of  $\beta$  to  $F_j$  is the map  $\theta$  defined in Lemma 3.7, it follows that  $\bar{a} \in F_j^p$ . Therefore there exists a subset  $\{j_1, \dots, j_s\} \subset [n]$  and positive integers  $t_k$  with  $t_k \leq r_{j_k}$  for  $k = 1, \dots, s$  such that  $P_{\bar{a}} = (x_{j_1 t_1}, \dots, x_{j_s t_s})$ . It is clear that  $P_{\bar{a}}$  is a prime ideal which contains  $I^p$  and  $\beta(a) = \bar{a}$ , where



$$a(i) = \begin{cases} t_k - 1, & \text{if } i = j_k \text{ for some } k, \\ \infty, & \text{otherwise.} \end{cases}$$

Now  $\bar{a} \in F^p$  if and only if  $P_{\bar{a}} \in \text{Min}(I^p)$ . Assume  $P_{\bar{a}} \notin \text{Min}(I^p)$ . Then there is a prime ideal  $Q \in \text{Min}(I^p)$  such that  $Q \subset P_{\bar{a}}$ . Suppose  $Q = (x_{i_1 e_1}, \dots, x_{i_h e_h})$ . Then  $\{i_1, \dots, i_h\} \subset \{j_1, \dots, j_s\}$  and  $\{e_1, \dots, e_h\} \subset \{t_1, \dots, t_s\}$ . On the other hand, since  $Q$  is a minimal prime ideal of  $I^p = \bigcap_{i=1}^t J_i^p$ , there exists an integer  $e \in [t]$  such that  $Q$  is one of the minimal prime ideals of

$$J_e^p = (x_{i_1}^{b_1}, \dots, x_{i_h}^{b_h})^p.$$

It follows that  $1 \leq e_i \leq b_i$  for  $i = 1, \dots, h$ . Therefore there exists  $b \in F_e$  with

$$b(i) = \begin{cases} e_k - 1, & \text{if } i \in \{i_1, \dots, i_h\}, \\ \infty, & \text{otherwise.} \end{cases}$$

This implies that  $a < b \leq m_e$ , and  $\text{infpt}(a) < \text{infpt}(b) = \text{infpt}(m_e)$ , a contradiction.

Next we show that  $\beta$  is injective: let  $a, b \in F$  and  $a \neq b$ . Then there exists an integer  $i$  such that  $a(i) \neq b(i)$ . We have to show  $\bar{a} \neq \bar{b}$ . We consider different cases:

- (i) If  $a(i) = 0$ , and  $b(i) \neq 0$ , then  $\bar{b}(i1) = \infty$  and  $\bar{a}(i1) = 0$ .
- (ii) If  $a(i) = \infty$ , and  $b(i) = t - 1$  where  $t \neq \infty$ , then  $\bar{a}(it) = \infty$  and  $\bar{b}(it) = 0$ .
- (iii) Suppose  $0 < t - 1 = a(i) \neq \infty$ . If  $b(i) = 0$ , then we have case (i). If  $b(i) = \infty$  then we have case (ii). Finally if  $0 < s - 1 = b(i) \neq \infty$ , then  $t \neq s$  since  $a(i) \neq b(i)$  and hence  $\bar{a}(it) = 0$  and  $\bar{b}(it) = \infty$ .

In all cases it follows that  $\bar{a} \neq \bar{b}$ .

Finally we show that  $\beta$  is surjective: let  $\bar{a} \in F^p \subset \bigcup_{i=1}^t F_i^p$  be any facet of  $\Gamma^p$ . Then there exists an integer  $i \in [t]$  such that  $\bar{a} \in F_i^p$ . Therefore  $P_{\bar{a}}$  is a minimal prime ideal of

$$J_i^p = (x_{i_1}^{a_1}, \dots, x_{i_k}^{a_k})^p,$$

and hence there exists  $t_i \leq a_i$  such that  $P_{\bar{a}} = (x_{i_1 t_1}, \dots, x_{i_k t_k})$ . Therefore

$$\bar{a}(ij) = \begin{cases} 0, & \text{if } i = i_r \text{ and } j = t_r \text{ for some } r \in [k], \\ \infty, & \text{otherwise.} \end{cases}$$

By our definition we have  $\bar{a} = \beta(a)$ , where  $a \in A$  with

$$a(i) = \begin{cases} t_r - 1, & \text{if } i = i_r \in \{i_1, \dots, i_k\}, \\ \infty, & \text{otherwise.} \end{cases}$$

It will be enough to show that  $a \in F$ . Since  $\bar{a} \in F_i^p$  and the restriction of  $\beta$  to  $F_i$  is a bijection from  $F_i$  to  $F_i^p$ , it follows that  $a \in F_i$ . If  $a \notin F$ , then there exists some  $j \neq i$ , such that  $a \leq m_j$ , and  $\text{infpt}(a) < \text{infpt}(m_j)$ . Therefore there exists an element  $b \in F_j$ , such that  $b(i) = a(i)$  for all  $i$  with  $b(i) \neq \infty$ . This implies that  $a < b$ , and  $\text{infpt}(a) < \text{infpt}(b) = \text{infpt}(m_j)$ . It follows from the definition of the map  $\beta$  that  $\bar{a} < \bar{b}$ , and that  $P_{\bar{b}}$  is a prime ideal with  $I^p \subset P_{\bar{b}} \subsetneq P_{\bar{a}}$ , a contradiction.  $\square$

Now let  $I \subset S$  be a monomial ideal and  $I^p \subset T$  be the polarization of  $I$ . Furthermore let

$$\pi : T \longrightarrow S, \quad x_{ij} \mapsto x_i.$$

be the epimorphism which attached to the polarization. Note that

$$\ker(\pi) = (x_{11} - x_{12}, \dots, x_{11} - x_{1r_1}, \dots, x_{n1} - x_{n2}, \dots, x_{n1} - x_{nr_n})$$

where  $r_i$  is the number of variables of the form  $x_{ij}$  which are needed for polarization. Set

$$y := x_{11} - x_{12}, \dots, x_{11} - x_{1r_1}, \dots, x_{n1} - x_{n2}, \dots, x_{n1} - x_{nr_n},$$

then  $y$  is a sequence of linear forms in  $T$ .

**Proposition 3.9.** *Let  $I \subset S$  be a monomial ideal and  $I^p$  be the polarization of  $I$ . Assume that*

$$\mathcal{G} : I^p = J_0 \subset J_1 \subset \dots \subset J_r = T$$

*is a clean filtration of  $I^p$ , and that*

$$\mathcal{F} : I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

*is the specialization of  $\mathcal{G}$ , that is,  $\pi(J_i) = I_i$  for all  $i$ . Then  $\mathcal{F}$  is a pretty clean filtration of  $I$  with  $I_k/I_{k-1} \cong S/\pi(Q_k)$ , where  $Q_k \cong J_k/J_{k-1}$ .*

*Proof.* For each  $k \in [r]$  the  $S$ -module  $I_k/I_{k-1}$  is a cyclic module since  $J_k/J_{k-1}$  is cyclic for all  $k$ . Let  $I_k/I_{k-1} \cong S/L_k$ , where  $L_k$  is a monomial ideal in  $S$ . It is clear that  $\pi(Q_k) \subset L_k$ . Indeed,  $Q_k = J_k/J_{k-1} : u_k$ , where  $J_k = (J_{k-1}, u_k)$  and where  $J_k/J_{k-1} \cong T/Q_k$ . If  $v \in Q_k$ , then  $vu_k \in J_{k-1}$ . It follows that  $\pi(vu_k) = \pi(v)\pi(u_k) \in \pi(J_{k-1}) = I_{k-1}$ , and hence  $\pi(v) \in I_{k-1} : \pi(u_k) = L_k$ .

We want to show that  $\pi(Q_k) = L_k$ .  $S$  and  $T$  are standard graded with  $\deg(x_i) = \deg(x_{ij}) = 1$  for all  $i$  and  $j$ , and  $\mathcal{G}$  is a graded prime filtration of  $I^p$ . Therefore  $\mathcal{F}$  is a graded filtration of  $I$ , and we have the following isomorphisms of graded modules  $J_i/J_{i-1} \cong T/Q_i(-a_i)$  and  $I_i/I_{i-1} \cong S/L_i(-a_i)$ , where  $a_i = \deg(u_i) = \deg(\pi(u_i))$ .

The filtrations  $\mathcal{G}$  and  $\mathcal{F}$  yield the following Hilbert series of  $T/I^p$  and  $S/I$ :

$$\text{Hilb}(T/I^p) = \sum_{i=1}^r \text{Hilb}(T/Q_i)t^{a_i} \quad \text{and} \quad \text{Hilb}(S/I) = \sum_{i=1}^r \text{Hilb}(S/L_i)t^{a_i}.$$

On the other hand since  $y$  is a regular sequence of linear forms on  $T/I^p$  and on  $T/Q_i$  for each  $i \in [r]$ , we have

$$\begin{aligned} \text{Hilb}(S/I) &= (1-t)^l \text{Hilb}(T/I^p) = (1-t)^l \sum_{i=1}^r \text{Hilb}(T/Q_i)t^{a_i} \\ &= \sum_{i=1}^r (1-t)^l \text{Hilb}(T/Q_i)t^{a_i} = \sum_{i=1}^r \text{Hilb}(S/\pi(Q_i))t^{a_i}, \end{aligned}$$

where  $l = |y|$ .

On the other hand, since  $\pi(Q_i) \subset L_i$ , we have the coefficientwise inequality  $\text{Hilb}(S/L_i) \leq \text{Hilb}(S/\pi(Q_i))$ , and equality holds if and only if  $L_i = \pi(Q_i)$ . Therefore we have

$$\text{Hilb}(S/I) = \sum_{i=1}^r \text{Hilb}(S/\pi(Q_i))t^{a_i} \geq \sum_{i=1}^r \text{Hilb}(S/L_i)t^{a_i} = \text{Hilb}(S/I).$$

It follows that  $L_i = \pi(Q_i)$  is a prime ideal for  $i = 1, \dots, r$ .

We know that  $\Gamma^p$  the multicomplex associated to  $I^p$  is shellable, since  $I^p$  is clean. Therefore we may assume that  $\mathcal{G}$  is obtained from a shelling of  $\Gamma^p$ . By [5, Corollary 10.7] and its proof (or directly from the definition of shellings of a simplicial complex) it follows that  $\mu(Q_i) \geq \mu(Q_{i-1})$  for all  $i \in [r]$ , where  $\mu(Q_i)$  is the number of generators of  $Q_i$ . Since by Corollary 3.6 each  $Q_i$  is of the form  $(x_{i_1 t_1}, \dots, x_{i_k t_k})$  with  $i_r \neq i_s$  for  $r \neq s$ , it follows that  $\mu(Q_i) = \mu(\pi(Q_i)) = \mu(L_i)$ . Therefore  $\mu(L_i) \geq \mu(L_{i-1})$  for all  $i$ . This implies that  $\mathcal{F}$  is a pretty clean filtration of  $S/I$ .  $\square$

As the main result of this section we have

**Theorem 3.10.** *Let  $I \subset S = K[x_1, \dots, x_n]$  be a monomial ideal and  $I^p$  its polarization. Then the following are equivalent*

- (a)  *$I$  is pretty clean.*
- (b)  *$I^p$  is clean.*

*Proof.* (a)  $\Rightarrow$  (b): Assume  $I$  is pretty clean. Then the multicomplex  $\Gamma$  associated with  $I$  is shellable. Let  $a_1, \dots, a_r$  be a shelling of  $\Gamma$ , and

$$\mathcal{F} : I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

the pretty clean filtration of  $I$  which is obtain from this shelling, i.e,  $I_i = \bigcap_{k=1}^{r-i} I(\Gamma(a_k))$ . Let  $\mathcal{F}^p$  be the polarization of  $\mathcal{F}$ . By Proposition 3.4,  $\mathcal{F}^p$  is a prime filtration of  $I^p$  with  $\ell(\mathcal{F}) = \ell(\mathcal{F}^p)$ . Using Proposition 3.8 we have

$$|F(\Gamma^p)| = |F(\Gamma)|.$$

On the other hand, since  $I$  is pretty clean we know that  $\ell(\mathcal{F}) = |F(\Gamma)|$ . Hence we conclude that

$$|F(\Gamma^p)| = \ell(\mathcal{F}^p).$$

Therefore, since  $\text{Min}(I^p) = \text{Ass}(I^p) \subset \text{Supp}(\mathcal{F}^p)$ , it follows that  $\text{Min}(I^p) = \text{Supp}(\mathcal{F}^p)$ , which implies that  $I^p$  is clean.

(b)  $\Rightarrow$  (a) follows from Proposition 3.9.  $\square$

As an immediate consequence we obtain the following result of [5, Corollary 10.7]:

**Corollary 3.11.** *Let  $I \subset S$  be a monomial ideal, and*

$$\mathcal{F} : I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

*a prime filtration of  $S/I$  with  $I_j/I_{j-1} \cong S/P_j$ . Then the following are equivalent:*

- (a)  *$\mathcal{F}$  is a pretty clean filtration of  $S/I$ .*
- (b)  *$\mu(P_i) \geq \mu(P_{i+1})$  for all  $i = 0, \dots, r-1$ .*

#### 4. A NEW CHARACTERIZATION OF PRETTY CLEAN MONOMIAL IDEALS

Let  $R$  be a Noetherian ring, and  $M$  a finitely generated  $R$ -module. For  $P \in \text{Spec}(R)$  the number  $\text{mult}_M(P) = \ell(H_P^0(M_P))$  is called the *length multiplicity* of  $P$  with respect to  $M$ .

Obviously, one has  $\text{mult}_M(P) > 0$  if and only if  $P \in \text{Ass}(M)$ . Assume now that  $(R, \mathfrak{m})$  is a local ring. Recall that the *arithmetic degree* of  $M$  is defined to be

$$\text{adeg}(M) = \sum_{P \in \text{Ass}(M)} \text{mult}_M(P) \cdot e(R/P),$$

where  $e(R/P)$  is the *multiplicity* of the associated graded ring of  $R/P$ .

First we notice the following

**Lemma 4.1.** *Suppose  $R$  is a Noetherian ring, and  $M$  a finitely generated  $R$ -module. Let*

$$\mathcal{F}: 0 = M_0 \subset M_1 \subset \dots \subset M_r = M$$

*be a prime filtration of  $M$  with  $M_i/M_{i-1} \cong R/P_i$ . Then*

$$\text{mult}_M(P) \leq |\{i \in [r-1] : M_{i+1}/M_i \cong R/P\}|$$

*for all  $P \in \text{Spec}(R)$ .*

*Proof.* If  $P \notin \text{Ass}(M)$ , the assertion is trivial. So now let  $P \in \text{Ass}(M)$ . Localizing at  $P$  we may assume that  $P$  is the maximal ideal of  $M$ .

Now we will prove the assertion by induction on  $\ell(\mathcal{F})$ . If  $\ell(\mathcal{F})=1$ , then the assertion is obviously true. Let  $\ell(\mathcal{F}) > 1$ . From the following short exact sequence

$$0 \rightarrow M_1 \rightarrow M \rightarrow M/M_1 \rightarrow 0$$

we get the following long exact sequence

$$0 \rightarrow H_P^0(M_1) \rightarrow H_P^0(M) \rightarrow H_P^0(M/M_1) \rightarrow \dots$$

Therefore  $\text{mult}_M(P) = \ell(H_P^0(M)) \leq \ell(H_P^0(M_1)) + \ell(H_P^0(M/M_1))$ . By induction hypothesis

$$\text{mult}_{M/M_1}(P) = \ell(H_P^0(M/M_1)) \leq |\{i \in [r-1] \setminus \{1\} : M_{i+1}/M_i \cong R/P\}|.$$

Now consider the following two cases:

(i) If  $M_1 \cong R/P$ , then  $\ell(H_P^0(M_1)) = 1$ . Therefore

$$\text{mult}_M(P) \leq 1 + \text{mult}_{M/M_1}(P) \leq |\{i \in [r-1] : M_{i+1}/M_i \cong R/P\}|.$$

(ii) If  $M_1 \not\cong R/P$ , then  $\ell(H_P^0(M_1)) = 0$ . Hence

$$\text{mult}_M(P) \leq |\{i \in [r-1] : M_{i+1}/M_i \cong R/P\}|.$$

□

Let  $S = K[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over the field  $K$ . Let  $I \subset S$  be a monomial ideal and  $\Gamma$  be the multicomplex associated to  $I$ . We denote the arithmetic degree of  $S/I$  by  $\text{adeg}(I)$ . Since  $e(S/P) = 1$  for all  $P \in \text{Ass}(I)$ , it follows that  $\text{adeg}(I) = \sum_{P \in \text{Ass}(I)} \text{mult}_I(P)$ , where  $\text{mult}_I(P) = \text{mult}_{S/I}(P)$ . By [13, Lemma 3.3]  $\text{adeg}(I) = |\text{Std}(I)|$ , where  $\text{Std}(I)$  is the set of standard pairs with respect to  $I$ . Also by [5, Lemma 9.14]  $|\text{Std}(I)| = |F(\Gamma)|$ . Since  $|F(\Gamma)| = |F(\Gamma^P)|$ , see 3.8, it follows that  $\text{adeg}(I) = \text{adeg}(I^P)$ , where  $I^P$  is the polarization of  $I$  and  $\Gamma^P$  the multicomplex associated to  $I^P$ .

In this part we want to show that  $\text{adeg}(I)$  is a lower bound for the length of any prime filtration of  $S/I$  and the equality holds if and only if  $S/I$  is a pretty clean module.

**Theorem 4.2.** *Let  $I \subset S$  be a monomial ideal and  $\mathcal{F}$  a prime filtration of  $I$ . One has*

- (1)  $\text{adeg}(I) \leq \ell(\mathcal{F})$ ;
- (2)  $\ell(\mathcal{F}) = \text{adeg}(I) \Leftrightarrow \mathcal{F}$  is a pretty clean filtration of  $I$ .

*Proof.* Part 1 is clear by Lemma 4.1.

One direction of (2) is [5, Corollary 6.4]. For the other direction assume  $\ell(\mathcal{F}) = \text{adeg}(I) = |F(\Gamma)| = |F(\Gamma^p)|$ . By Proposition 3.4  $\mathcal{F}^p$  is a prime filtration of  $I^p$  with  $\ell(\mathcal{F}^p) = |F(\Gamma^p)| =$  the number of minimal prime ideals of  $\Gamma^p$ . Therefore  $\mathcal{F}^p$  is a clean filtration of  $I^p$ , so by Theorem 3.10  $\mathcal{F}$  is a pretty clean filtration of  $I$ .  $\square$

Combining Theorem 4.2 with Theorem 3.10 we get

**Corollary 4.3.** *Let  $I \subset S$  be a monomial ideal. Assume  $\Gamma$  is the multicomplex associated to  $I$  and  $I^p$  the polarization of  $I$ . The following are equivalent:*

- (a)  $\Gamma$  is shellable;
- (b)  $I$  is pretty clean;
- (c) There exists a prime filtration  $\mathcal{F}$  of  $I$  with  $\ell(\mathcal{F}) = \text{adeg}(I)$ ;
- (d)  $I^p$  is clean;
- (e) If  $\triangle$  be the simplicial complex associated to  $I^p$ , then  $\triangle$  is shellable.

If  $R$  is a Noetherian ring and  $M$  a finitely generated  $R$ -module with pretty clean filtration  $\mathcal{F}$ , then  $\text{Ass}(M) = \text{Supp}(\mathcal{F})$ , see [5, Corollary 3.6]. The converse is not true in general as shown by an example in [5]. The example given there is a cyclic module defined by a non-monomial ideal. The following example shows that even in the monomial case the converse does not hold in general.

**Example 4.4.** Let  $S = K[a, b, c, d]$  be the polynomial ring over the field  $K$ ,  $I \subset S$  the ideal

$$I = (a, b) \cdot (c, d) \cdot (a, c, d) = (abc, abd, acd, ad^2, a^2d, ac^2, a^2c, bcd, bc^2, bd^2)$$

and  $M = S/I$ . We claim that the module  $M = S/I$  is not pretty clean, but that  $M$  has a prime filtration  $\mathcal{F}$  with  $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$ .

Note that  $(a, b) \cap (c, d) \cap (a, c, d^2) \cap (a, c^2, d) \cap (a^2, b, c, d^2) \cap (a^2, b, c^2, d)$  modulo  $I$  is an irredundant primary decomposition of  $(0)$  in  $M$ .

We see that  $\text{Ass}(M) = \{(a, b), (c, d), (a, c, d), (a, b, c, d)\}$ . It is clear that

$$\begin{aligned} \mathcal{F} : I &= I_0 \subset I_1 = (I, ac) \subset I_2 = (I_1, ad) \subset I_3 = (I_2, bd) \\ &\subset I_4 = (I_3, bc) \subset I_5 = (I_4, a) \subset I_6 = (a, b) \subset S \end{aligned}$$

is a prime filtration of  $M$  with  $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$ . Indeed  $I_1/I \cong I_2/I_1 \cong S/(a, b, c, d)$ ,  $I_3/I_2 \cong I_4/I_3 \cong I_6/I_5 \cong S/(a, c, d)$  and  $I_5/I_4 \cong S/(c, d)$ .

From the above irredundant primary decomposition of  $I$  it follows that  $\text{adeg}(I)=6$ . But the length of any prime filtration of  $I$  is at least 7. Therefore  $I$  can not be pretty clean. In other words, from [5, Corollary 1.2] it follows that  $D_1(M) = ((a, b) \cap (c, d))/I$  and that  $D_2(M) = M$ , where  $D_i(M)$  is the largest submodule of  $M$  with  $\dim(M) \leq i$ , for  $i = 0, \dots, \dim(M)$ . It follows that  $D_2(M)/D_1(M) \cong S/(a, b) \cap (c, d)$  is not clean. Knowing now  $D_2(M)/D_1(M)$  is not clean, we conclude from [5, Corollary 4.2] that  $M = S/I$  is not pretty clean.

## REFERENCES

- [1] J. Apel, On a conjecture of R. P. Stanley; Part I-Monomial Ideals, *J. of Alg. Comb.* **17**, (2003), 36–59.
- [2] W. Bruns, J. Herzog, *Cohen Macaulay rings*, Revised Edition, Cambridge, 1996.
- [3] A. Dress, A new algebraic criterion for shellability, *Beitrage zur Alg. und Geom.*, **340**(1), (1993), 45–55.
- [4] D. Eisenbud, *Commutative algebra, with a view towards geometry*, Graduate Texts Math. Springer, 1995.
- [5] J. Herzog, D. Popescu, Finite filtrations of modules and shellable multicomplexes, *math. AC/0502282*.
- [6] S. Hosten, R. R. Thomas, Standard pairs and group relaxations in integer programming, *J. Pure Appl. Alg.* **139**, (1999), 133–157.
- [7] D. MacLagan, G. Smith, Uniform bounds on multigraded regularity, *J. Alg. Geom.* **14**, (2005), 137–164.
- [8] H. Masumura, *Commutative Ring theory*, Cambridge, 1986.
- [9] P. Schenzel, On the dimension filtration and Cohen-Macaulay filtered modules, *Proceed. of the Ferrara meeting in honour of Mario Fiorentini*, ed. F. Van Oystaeyen, Marcel Dekker, New-York, 1999.
- [10] R. S. Simon, Combinatorial Properties of “Cleanness”, *J. of Alg.* **167**, (1994), 361–388.
- [11] R. P. Stanley, *Combinatorics and Commutative Algebra*, Birkhäuser, 1983.
- [12] R. P. Stanley, Linear Diophantine equations and local cohomology, *Invent. Math.* **68**, (1982), 175–193.
- [13] B. Sturmfels, N. V. Trung, W. Vogel, Bounds on Degrees of Projective Schemes, *Math. Ann.* **302**, (1995), 417–432.
- [14] R. H. Villarreal, *Monomial Algebras*, Dekker, NewYork, NY, 2001.

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